# Active Fault Tolerant Control for Nonlinear Systems with Simultaneous Actuator and Sensor Faults

**Montadher Sami, Ron J Patton**

**Abstract:** The goal of this paper is to describe a novel fault tolerant tracking control (FTTC) strategy based on robust fault estimation and compensation of simultaneous actuator and sensor faults. Within the framework of fault tolerant control (FTC) the challenge is to develop an FTTC design strategy for nonlinear systems to tolerate simultaneous actuator and sensor faults that have bounded first time derivatives. The main contribution of this paper is the proposal of a new architecture based on a combination of actuator and sensor Takagi-Sugeno (T-S) proportional state estimators augmented with proportional and integral feedback (PPI) fault estimators together with a T-S dynamic output feedback control (TSDOFC) capable of time-varying reference tracking. Within this architecture the design freedom for each of the T-S estimators and the control system are available separately with an important consequence on robust $L_2$ norm fault estimation and robust $L_2$ norm closed-loop tracking performance. The FTTC strategy is illustrated using a nonlinear inverted pendulum example with time-varying tracking of a moving linear position reference.

**Keywords**: Active fault tolerant control, Dynamic output feedback control, LMI formulation, Fault estimation, Tracking control, T-S fuzzy systems.

## 1. INTRODUCTION

Due to the increased demand for maintaining required system performances in different conditions of operation, there has been a rapidly growing interest in the field of FTC in the last two decades [1-3]. As a specific defintion the FTC system is control system (or control loop) that has the ability to maintain required system closed performance and stability (within acceptable degradation) even if faults occur in different parts of the system under control. In this way the FTC system is said to tolerate faults acting within the control system feedback structure. Following this, FTC includes most attributes of robust control and is a wider subject in which the robustness is extended to encompass faults as well is exogenous disturbance and modelling uncertainty. Comprehensive surveys describing research activities on FTC methods covering the last 15 years are to be found in [4, 5].

Traditionally safety critical systems have provided much of the main motivation for the development of the subject of FTC, however research during the last decade has shown that FTC methods represent promising approaches to handle several practical fault scenarios for real system applications. For example, in [6], FTC is utilised to compensate the effect of existing friction in mechatronic systems. In [7] FTC is used to enhance the performance of electromagnetic suspension system through tolerating the effect of air gap sensor fault and an accelerometer fault. In [8] a way of tolerating the effect of a faulty thruster is proposed through reallocation of thruster forces of an autonomous underwater vehicle. [9] describes the application of FTC methods for flight control of unmanned airborne vehicles. Recently FTC has been considered as a viable approach to ensure offshore wind turbine sustainability in terms of power maximization and fault tolerance [10, 11].

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The recent FTC literature shows a steady increase in interest of developing strategies for nonlinear systems [12-15]. The last five years have witnessed the publication of a number of research studies on FTC that make use of multiple-model strategies for nonlinear systems that can be used for both modelling and control, for example via the use of T-S fuzzy inference modelling [16-18]. T-S models are usually considered for fault estimation in nonlinear systems since the approach provides an opportunity to handle system nonlinearity via well developed modern linear systems optimisation and design tools. However, most of the work on this subject employing T-S fault estimation are focussed on the actuator fault estimation and compensation problem [16-18] and do not consider simultaneous actuator and sensor faults.

The paper presents a new approach for AFTC for nonlinear systems with simultaneously acting actuator and sensor faults. The ideas focus on the design of an observer-based fault tolerant tracking controller for nonlinear systems described in T-S model form. It is assumed that the system is affected by both sensor and actuator faults simultaneously tolerated using the fault estimation and compensation concept.

The proposed strategy involves the design of (i) a TSDOFC responsible for minimising the tracking error between the reference and system output signals during nominal operation, and (ii) two T-S fuzzy observers dedicated to provide separate estimates of the actuator and sensor faults for the purpose of fault compensation. The nonlinear example of an inverted pendulum with time-varying cart position reference is used to illustrate the proposed FTTC strategy. Both additive and parametric fault scenarios are considered for simultaneous actuator and sensor faults. The tracking system is introduced to induce significant nonlinearity in the inverted pendulum system.

The paper is organised as follows. Section 2 outlines the advantages and limitations of the use of a fault estimation and compensation strategy for AFTC and based on this the proposed methodology is outlined and a suitable architecture is given. Section 3 enters into a description of the observer based FTTC approach followed by three subsections illustrating the stability and performance design conditions for (i) sensor fault estimate observer, (ii) the actuator fault estimate observer, and (iii) the TSDOFC. In Section 4, the results are given, using the inverted pendulum example to illustrate the importance and effectiveness of the proposed FTTC scheme with simultaneous actuator and sensor faults. Section 5 provides a concluding discussion.

## 2. THE PROPOSED ARCHITECTURE FOR ACTIVE FTC

The goal of this work is to develop a novel FTC strategy based on robust fault estimation and compensation of simultaneous actuator faults ($f_a$) and sensor faults ($f_s$) to maintain the performance and stability of the "baseline" or nominal control system during both faulty and fault-free cases. An FTC scheme is proposed that is based on the combination of (a) robust control and (b) independent estimates of the each of the actuator faults ($\hat{f}_a$) and sensor faults ($\hat{f}_s$). The controller is required to be robust against expected actuator and sensor fault estimation errors as well as the bounded reference signal. It is clear from the architecture shown in Fig. 1 that the scheme includes dedicated fault estimation observers in order to ensure accurate estimation and compensation of each of the actuator and sensor faults. Moreover, as the accuracy of fault estimation is of paramount importance the original PPI observer formulation of [19] has been extended to the multi-model case within a T-S fuzzy inference modelling structure. In the sensor fault estimator design the actuator fault signal is considered as an *unknown input* signal that can be compensated directly in the sensor fault estimation. Conversely, the effect of the sensor fault can be compensated in the estimated actuator fault. Hence, based on the architecture of Fig. 1 the actuator and sensor fault estimation errors must each be bounded (as well as the first time derivative of each fault).

***Remark 1***: AFTC systems in general are designed to handle the occurrence of system faults in real-time by using fault estimation/compensation methods, adaptive control or controller reconfiguration mechanisms with various advantages and disadvantages. Fault estimation and compensation methods obviate the need for the use of an FDI unit. However, the performance of these methods

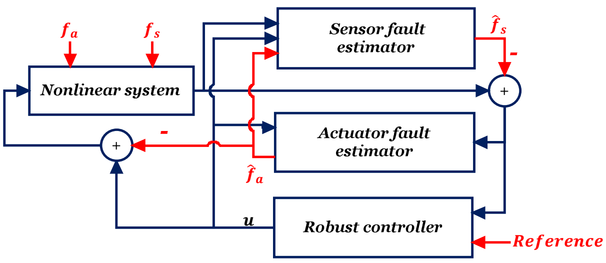


Fig. 1 Proposed AFTC system architecture

is highly affected by fault estimation accuracy and the presence of any simultaneous faults. On the other hand, the philosophy of adaptive control fits well with the AFTC approach due to the ability of adaptive control systems to adjust controller parameters online based on measured signals [20-23]. Clearly, the use of adaptive control methods as an approach to AFTC obviates the need for FDI. However, in this method, sensor faults represent the most challenging of fault scenarios for AFTC and have rarely been considered in fault tolerant adaptive control methods. For example, output feedback adaptive tracking control can tolerate actuator and/or system faults. On the other hand if sensor faults have occurred the adaptation will force the faulty output to follow the reference signals and hence the control signal will no longer be suitable for the system under control. The CR-based FTC approach can handle more general faults and or failure cases through either off-line or online variation of the structure and/or the parameters of the controller based on the information delivered from an FDI unit. However, the main challenge of this method is that the time required to reconfigure the control system must be as low as possible. This is important in practice where the time windows during which the system remains stabilisable in the presence of a fault are very short[24, 25].

Based on Remark 1, the main motivation to use the fault estimation and compensation approach is to overcome the reconfiguration time problem arising in CR-based FTC. Furthermore, the inability of adaptive control-based FTC to tolerate sensor faults means that fault estimation and compensation represents the all round most appropriate method for the sensor fault case of FTC.

Section 3 deals with the extension of the above concept to include a TSDOFC controller as a special case of the robust baseline controller of Section 2 (see Fig. 1) and develops the theory for applying the proposed strategy to nonlinear systems described via T-S fuzzy models.

## 3. ACTIVE FTC FOR NONLINEAR SYSTEMS VIA T-S FUZZY MODELLING

This section describes the proposed strategy for active actuator and sensor fault tolerant TSDOFC. The TSDOFC is designed to force specific outputs to follow a given reference input (in both faulty and fault-free cases) with robustness against exogenous inputs/outputs (actuator/sensor fault estimation error). The T-S PPI observers are used as a form of analytical redundancy responsible for robustly compensating the effects of actuator and sensor faults from the system inputs and outputs and hence ensure the robustness of the overall closed-loop system.

The T-S fuzzy controller and T-S fuzzy observer designs are model-based methods and hence the first design step involves the derivation of the fuzzy model of the plant corresponding to different operating conditions. In the fuzzy control design, each "control rule" is designed from the corresponding rule of a T-S fuzzy model and the fuzzy controller design problem is to determine the local feedback gains within a *parallel distributed compensation* structure [26]. Although the fuzzy controller is constructed using the local design structure, the feedback gains should be determined using global design conditions. It is reported in [26] that for the fuzzy state estimate feedback "*the global design conditions are needed to guarantee the global stability and control performance*". Hence, the fuzzy control designer does not have freedom to assign the local system closed-loop poles anywhere in the stable complex plane. Therefore, the observer based T-S state feedback control system suffers a major drawback in that the observer dynamics may not be assigned freely to satisfy closed-loop performance requirements. The TSDOFC has been proposed in this strategy to overcome the limitation of T-S observer-based state feedback control.

To illustrate the basic idea of representing a dynamic system in T-S fuzzy model form consider the following general form of a nonlinear system with no exogenous inputs, i.e. disturbances or faults.:

$$\left.\begin{aligned} \dot{x} &= f\left(x(t), u(t)\right) \\ y &= g\left(x(t)\right) \end{aligned}\right\} \quad (1)$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^m$ is the input vector and $y(t) \in \mathcal{R}^l$ is the output vector. The mathematical representation, i.e. fuzzy nonlinear approximation, for the system is given as follows in terms of blending of appropriate local model systems:

$$\left.\begin{aligned} \dot{x} &= A(p)x + B(p)u(t) \\ y &= C(p)x \end{aligned}\right\} \quad (2)$$

The matrices $A(p) \in \mathcal{R}^{n*n}(=\sum_{i=1}^{r} h_i(p)A_i)$, $B(p) \in \mathcal{R}^{n*m}(=\sum_{i=1}^{r} h_i(p)B_i)$ and $C(p) \in \mathcal{R}^{l*n}(=\sum_{i=1}^{r} h_i(p)C_i)$, are the known system matrices, $r$ is the number of fuzzy rules and the term $h_i(p)$ is the weighting function that depends on the variable that assumed to be measured, the so-called "premise variable $(p)$ " (or scheduling variable). The weighting function must satisfy the following properties for all time t.

$$\sum_{i=1}^{r} h_i(p) = 1 \ ; \ h_i(p) \geq 0, \quad \mathrm{i} = 1, 2, ..., \mathrm{r} \quad (3)$$

where $p = \left[p_1, \ldots, p_p\right]$, $w_i(p) = \prod_{j=1}^{p} M_{ij}\left(p_j\right)$

$$h_i(p) = \frac{w_i(p)}{\sum_{i=1}^{r} w_i(p)}$$

the term $M_{ij}(p_j)$ is the grade of membership of $p_j$ in $M_{ij}$

### 3.1 Sensor fault hiding observer design

Consider a T-S fuzzy model with actuator and sensor fault signals, described as follows:

$$\left.\begin{aligned} \dot{x} &= A(p)x + B(p)(u + f_a) \\ y &= Cx + D_f f_s \end{aligned}\right\} \quad (4)$$

where the matrices $A(p)$ and $B(p)$ are as defined in Eq.(2), $C \in \mathcal{R}^{l*n}$is the output matrix, $D_f \in R^{l*g}$ is known matrix, $f_s \in R^g$, and $f_a \in R^m$ are sensor and actuator faults, respectively.

To avoid the direct multiplication of the sensor and/or noise by the observer gain, an augmented system state with output filter states is constructed. The filtered output is given as follows:

$$\dot{x}_s = -A_s x_s + A_s Cx + A_s D_f f_s \quad (5)$$

where $-A_s \in R^{l*l}$ is a stable matrix. The augmented state system is given as:

$$\left.\begin{aligned} \dot{\bar{x}} &= \bar{A}(p)\bar{x} + \bar{B}(p)(u + f_a) + \bar{D}_f f_s \\ \bar{y} &= \bar{C}\,\bar{x} \end{aligned}\right\} \quad (6)$$

$$\bar{A}_i = \begin{bmatrix} A(p) & 0 \\ A_s C & -A_s \end{bmatrix}; \bar{B}_i(p) = \begin{bmatrix} B(p) \\ 0 \end{bmatrix}; \bar{D}_f = \begin{bmatrix} 0 \\ A_s D_f \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} x \\ x_s \end{bmatrix}; \bar{C} = \begin{bmatrix} 0 & I_l \end{bmatrix}$$

As illustrated in Fig. 1 the proposed control strategy requires the estimation of the fault effects on the closed-loop system. To ensure ability to deal with time-varying fault scenarios for which the first time derivative of each fault is assumed bounded, T-S fuzzy proportional state estimators augmented with proportional and integral feedback (PPI) for sensor fault estimation are designed as an extension to the work of [19] for linear systems.

***Remark 2:***

- In the literature, several FTC strategies have been proposed under the constraint that fault signals are constant (i.e. the fault first time derivative is zero) [16, 17]. The work in this paper focuses on the design of an AFTC system that has the capability of tolerating wide range of fault scenarios (i.e. time varying faults).
- The first stage of the development of a fault-tolerant system requires Failure Mode and Effective Analysis (FMEA) which aims to provide a complete coverage of possible occurring faults in the closed-loop as well as the corresponding remedial measures [1, 27]. Hence, information about fault time varying behaviour (and hence fault time derivative) can be anticipated during FMEA. This FMEA phase of FTC system design is beyond the scope of this paper.

Following the work of [19] assume that the signal $(\dot{f}_s)$ is bounded. Then the following fuzzy observer is proposed to simultaneously estimate the system states and sensor fault:

$$\left.\begin{aligned} \dot{\hat{\bar{x}}} &= \bar{A}(p)\hat{\bar{x}} + \bar{B}(p)(u + \hat{f}_s) + \bar{D}_f \hat{f}_s + \bar{L}(p)(\bar{C}\bar{x} - \bar{C}\hat{\bar{x}}) \\ \hat{f}_s(t) &= F(p)\bar{C}(e_x + \dot{e}_x) \end{aligned}\right\} \quad (7)$$

where $\hat{\bar{x}} \in \mathcal{R}^{n+l}$ is the estimation of the state vector $\bar{x}$, $\hat{f}_a$ is the actuator fault estimate delivered by the other observer, $\bar{L}(p) \in \mathcal{R}^{(n+l)\times l}$ and $F(p) \in \mathcal{R}^{g\times l}$ are the observer gains to be designed, and $e_x$ is the state estimation error defined as:

$$e_x = \bar{x} - \hat{\bar{x}} \tag{8}$$

The state estimation error dynamics are then given as:

$$\dot{e}_x = \left(\bar{A}(p) - \bar{L}(p)\bar{C}\right)e_x + \bar{D}_f e_{f_s} + \bar{B}(p)e_{f_a} \tag{9}$$

where $e_{f_s} \in \mathcal{R}^g$ and $e_{f_a} \in \mathcal{R}^m$ are the sensor and actuator fault estimation errors defined as:

$$\left.\begin{aligned} e_{f_s} &= f_s - \hat{f}_s \\ e_{f_a} &= f_a - \hat{f}_a \end{aligned}\right\} \tag{10}$$

Using (7) and (9) the fault estimation error dynamics are as follows:

$$\begin{aligned} \dot{e}_{f_s} = \dot{f}_s - F(p)\bar{C}\left(\bar{A}(p) - \bar{L}(p)\bar{C} + I\right)e_x - \\ F(p)\bar{C}\bar{D}_f e_{f_s} - F(p)\bar{C}\bar{B}(p)e_{f_a} \end{aligned} \tag{11}$$

The augmented estimator will then be of the following form:

$$\dot{\tilde{e}}_{as}(t) = \tilde{A}_s(p,p)\tilde{e}_{as} + \tilde{N}(p,p)\tilde{z} \tag{12}$$

$$\tilde{A}_s(p,p) = \begin{bmatrix} \bar{A}(p) - \bar{L}(p)\bar{C} & \bar{D}_f \\ -F(p)\bar{C}\left(\bar{A}(p) - \bar{L}(p)\bar{C} + I\right) & -F(p)\bar{C}\bar{D}_f \end{bmatrix}$$

$$\tilde{e}_{as} = \begin{bmatrix} e_x \\ e_{f_s} \end{bmatrix}, \tilde{N}(p,p) = \begin{bmatrix} \bar{B}(p) & 0 \\ -F(p)\bar{C}\bar{B}(p) & I \end{bmatrix}, \tilde{z} = \begin{bmatrix} e_{f_a} \\ \dot{f}_s \end{bmatrix}$$

The objective now is to compute the gains $\bar{L}(p)$ and $F(p)$ such that exogenous input $\tilde{z}$ in (12) is attenuated below the desired level $\gamma$ to ensure robust regulation performance. The location of the closed-loop system poles affect the estimation transient response. Hence, the sensor fault observer can be designed to constrain the estimation error system eigenvalues to lie globally in a complex region. This is defined by merging different eigenvalue constraints to produce a $\mathcal{D}(\rho,\alpha,\beta,\theta)$ LMI region in which the vertical line at $\rho$ bounds the stability region, where $\alpha$ and $\beta$ are the radius and centre of the disc region, and $\theta$ is the angle of sector of the $\alpha$ and $\beta$ circle (see [28] for more details).

***Theorem 1.*** *The eigenvalues of the estimation error dynamics are located in a LMI region in the complex plane defined by $\mathcal{D}(\rho,\alpha,\beta,\theta)$, and the error dynamics are stable and the $H_\infty$ performance is guaranteed with an attenuation level $\gamma$, (provided that the signal ($\tilde{z}$) is bounded), if there exists a SPD matrix $P_1$, and matrices $H_i, F_i$, and scalar parameters $\mu$, $\rho, \alpha, \beta$, and $\theta$ satisfying the following LMI constraints:*

*Minimize $(\gamma + \mu)$ such that:*

$$\left.\begin{aligned} &\Sigma_i + \Sigma_i^T + 2\rho\bar{P} < 0 \\ &\begin{bmatrix} -\alpha\bar{P} & \beta\bar{P} + \Sigma_i \\ \left(\beta\bar{P} + \Sigma_i\right)^T & -\alpha\bar{P} \end{bmatrix} < 0 \\ &\begin{bmatrix} sin(\theta)\left[\Sigma_i + \Sigma_i^T\right] & cos(\theta)\left[\Sigma_i + \Sigma_i^T\right] \\ cos(\theta)\left[\Sigma_i + \Sigma_i^T\right] & sin(\theta)\left[\Sigma_i + \Sigma_i^T\right] \end{bmatrix} < 0 \end{aligned}\right\} \tag{13}$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & 0 & C_{p1}^T & 0 \\ * & \Psi_{22} & \Psi_{23} & I & 0 & C_{p2}^T \\ * & * & -\gamma I & 0 & 0 & 0 \\ * & * & * & -\gamma I & 0 & 0 \\ * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & -\gamma I \end{bmatrix} < 0 \tag{14}$$

$$\begin{bmatrix} \mu I & \bar{D}_f^T P_1 - F(p)\bar{C} \\ * & \mu I \end{bmatrix} > 0 \tag{15}$$

$$\bar{L}(p) = P_1^{-1}\bar{H}(p)$$

$$\Psi_{11} = P_1\bar{A}(p) + \left(P_1\bar{A}(p)\right)^T - \bar{H}(p)\bar{C} - \left(\bar{H}(p)\bar{C}\right)^T$$

$$\Psi_{12} = -\left(\bar{A}^T(p)P_1\bar{D}_f - \bar{C}^T\bar{H}^T(p)\bar{D}_f\right); \Psi_{13} = P_1\bar{B}(p)$$

$$\Psi_{22} = -2\bar{D}_f^T P_1\bar{D}_f; \Psi_{23} = -\bar{D}_f^T P_1\bar{B}(p)$$

$$\Sigma_i = \bar{P}\tilde{A}(p,p) = \begin{bmatrix} P_1\bar{A}(p) - \bar{H}(p)\bar{C} & P_1\bar{D}_f \\ -\left(\bar{A}^T(p)P_1\bar{D}_f - \bar{C}^T\bar{H}^T(p)\bar{D}_f + \bar{D}_f^T P_1\right)^T & -\bar{D}_f^T P_1\bar{D}_f \end{bmatrix}$$

**Proof:** Let the performance output be defined as follows:

$$\tilde{e}_p = \bar{C}_p\tilde{e}_{as} \text{ where } \bar{C}_p = \begin{bmatrix} C_{p1} & 0 \\ 0 & C_{p2} \end{bmatrix}$$

$C_{p1} \in \mathcal{R}^{k*n}$ and $C_{p2} \in \mathcal{R}^{g*g}$. The estimation performance objective can now be defined as [29]:

$$\frac{\left\|\tilde{e}_p\right\|_2}{\left\|\tilde{z}\right\|_2} \le \gamma = \frac{1}{\gamma}\int_0^\infty \tilde{e}_{as}^T\bar{C}_p^T\bar{C}_p\tilde{e}_{as}dt - \gamma\int_0^\infty \tilde{z}^T\tilde{z} \le 0 \tag{16}$$

Now consider the following candidate Lyapunov function for the augmented system (12)

$$\upsilon\left(\tilde{e}_{as}\right) = \tilde{e}_{as}^T\bar{P}\tilde{e}_{as} \quad , where\ \bar{P} > 0 \tag{17}$$

To achieve the required performance (16) and stability of the augmented system (12) the following inequality should hold:

$$\dot{\upsilon}\left(\tilde{e}_{as}\right) + \frac{1}{\gamma}\tilde{e}_p^T\tilde{e}_p - \gamma\tilde{z}^T\tilde{z} < 0 \tag{18}$$

where $\dot{\upsilon}(\tilde{e}_{as})$ is the derivative of candidate Lyapunov function ( $\upsilon(\tilde{e}_{as}) = \tilde{e}_{as}^T\bar{P}\tilde{e}_{as}$ ) in terms of Eq.(12). Inequality (18) can now be re-written as:

$$\begin{aligned} \dot{\upsilon}(\tilde{e}_{as}) = \tilde{e}_{as}^T\left(\tilde{A}_s^T(p,p)\bar{P} + \bar{P}\tilde{A}_s(p,p)\right)\tilde{e}_{as} + \\ \tilde{e}_{as}^T\bar{P}\tilde{N}(p,p)\tilde{z} + \tilde{z}^T\tilde{N}^T(p,p)\bar{P}\tilde{e}_{as} \end{aligned} \tag{19}$$

By using (19), and the Schur Complement Theorem, then inequality (18) implies that the following inequality must hold:

$$\begin{bmatrix} \tilde{A}_s^T(p,p)\bar{P}+\bar{P}\tilde{A}_s(p,p) & \bar{P}\tilde{N}(p,p) & \bar{C}_{p1}^T & 0 \\ * & -\gamma I & 0 & \bar{C}_{p2}^T \\ * & * & -\gamma I & 0 \\ * & * & * & -\gamma I \end{bmatrix} < 0 \quad (20)$$

To conform to the format of (12) $\bar{P}$ is structured as follows:

$$\bar{P} = \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix} > 0 \quad (21)$$

By substituting the corresponding values of $\bar{P}, \tilde{A}_s(p,p)$, $\tilde{N}(p,p)$ and using the variable change $\bar{H}(p) = P_1\bar{L}(p)$, and the equality

$$F(p)\bar{C} = \bar{D}_f^T P_1 \quad (22)$$

The LMI in (14) is obtained.

***Remark 3:*** The equality (22) can be relaxed using the following optimization problem [19]

minimise $\mu$ such that

$$\begin{bmatrix} \mu I & \bar{D}_f^T P_1 - F(p)\bar{C} \\ * & \mu I \end{bmatrix} > 0 \quad (23)$$

To prove the validity of the LMI (13) the following Lemma [28] is required:

*Lemma 1* [28]: For the control problem the matrix $\mathcal{A}$ is $\mathcal{D}(\rho,\alpha,\beta,\theta)$-stable if and only if there exists a symmetric matrix $\mathcal{X} > 0$ such that:

$$\left.\begin{array}{c} \mathcal{A}\mathcal{X}+(\mathcal{A}\mathcal{X})^T+2\rho\mathcal{X}<0 \\ \begin{bmatrix} -\alpha\mathcal{X} & \beta\mathcal{X}+\mathcal{A}\mathcal{X} \\ \beta\mathcal{X}+(\mathcal{A}\mathcal{X})^T & -\alpha\mathcal{X} \end{bmatrix}<0 \\ \begin{bmatrix} \sin(\theta)\left[\mathcal{A}\mathcal{X}+(\mathcal{A}\mathcal{X})^T\right] & \cos(\theta)\left[\mathcal{A}\mathcal{X}-(\mathcal{A}\mathcal{X})^T\right] \\ \cos(\theta)\left[\mathcal{A}\mathcal{X}-(\mathcal{A}\mathcal{X})^T\right] & \sin(\theta)\left[\mathcal{A}\mathcal{X}+(\mathcal{A}\mathcal{X})^T\right] \end{bmatrix}<0 \end{array}\right\} \quad (24)$$

To ensure the suitability of the LMI constraints given in Lemma1 for observer design constraints, the three inequalities of (24) have been rewritten in the following *observer equivalent form*:

$$\left.\begin{array}{c} \mathbb{X}\mathcal{A}+(\mathbb{X}\mathcal{A})^T+2\rho\mathbb{X}<0 \\ \begin{bmatrix} -\alpha\mathbb{X} & \beta\mathbb{X}+\mathbb{X}\mathcal{A} \\ \beta\mathbb{X}+(\mathbb{X}\mathcal{A})^T & -\alpha\mathbb{X} \end{bmatrix}<0 \\ \begin{bmatrix} \sin(\theta)\left[\mathbb{X}\mathcal{A}+(\mathbb{X}\mathcal{A})^T\right] & \cos(\theta)\left[\mathbb{X}\mathcal{A}-(\mathbb{X}\mathcal{A})^T\right] \\ \cos(\theta)\left[\mathbb{X}\mathcal{A}-(\mathbb{X}\mathcal{A})^T\right] & \sin(\theta)\left[\mathbb{X}\mathcal{A}+(\mathbb{X}\mathcal{A})^T\right] \end{bmatrix}<0 \end{array}\right\}$$

where $\mathbb{X} > 0$.

**Proof:** By using the Congruence Lemma [30], pre- and post-multiply the first LMI of (24) by $\mathcal{X}^{-1}$, and pre and post multiply the other two constraints by $diag(\mathcal{X}^{-1},\mathcal{X}^{-1})$. Hence, the observer equivalent form of LMI constraints (24) is obtained after using the change of variables $\mathbb{X} = \mathcal{X}^{-1}$. This completes the proof.

Based on Lemma 1, inequality (13) can be obtained after substituting $\mathbb{X} = \bar{P} = \begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix} > 0$, and $A = \tilde{A}(p,p)$ as defined in (12). This completes the proof of Theorem 1.

### 3.2 Actuator fault estimate observer design

This subsection considers the actuator fault estimator design, with the observer driven by the corrected (sensor fault compensated) output and control signals (see Fig. 2). Therefore, the system given in Eq.(4) becomes:

$$\left.\begin{array}{l} \dot{x} = A(p)x + B(p)(u+f_a) \\ y = Cx + D_f e_{f_s} \end{array}\right\} \quad (25)$$

Based on the same arguments as given in *Section 3.1*, the T-S fuzzy PPI observer is used for estimating the actuator fault.

Under the assumption that the actuator fault first time derivative and sensor fault estimation error $(\dot{f}_a, e_{f_s})$ are bounded, then the following T-S fuzzy observer is proposed to simultaneously estimate the system states and actuator fault:

$$\left.\begin{array}{l} \dot{\hat{x}} = A(p)\hat{x} + B(p)\left(u+\hat{f}_a\right) + L_a(p)\left(Cx + D_f e_{f_s} - C\hat{x}\right) \\ y = Cx + D_f e_{f_s} \end{array}\right\} \quad (26)$$

where $\hat{x} \in \mathcal{R}^n$ is the estimation of the state vector $x$, $L_a(p) \in \mathcal{R}^{n\times l}$, and $F_a(p) \in \mathcal{R}^{m\times l}$ are the observer gains to be designed, and $e_x$ is the state estimation error. The state estimation error dynamic then:

$$\dot{e}_x = (A(p) - L_a(p)C)e_x + B(p)e_{f_a} - L_a(p)D_f e_{f_s} \quad (27)$$

Using (26) and (27) the fault estimation error dynamics are as follows:

$$\dot{e}_{f_a} = \dot{f}_a - F_a(p)C(A(p) - L_a(p)C + I)e_x - F_a(p)CB(p)e_{f_a} + F_a(p)CL_a(p)D_f e_{f_s} \quad (28)$$

The augmented estimator will then be of the following form:

$$\dot{\tilde{e}}_a(t) = \tilde{A}(p,p)\tilde{e}_a + \tilde{N}(p,p)\tilde{z} \quad (29)$$

$$\tilde{A}(p,p) = \begin{bmatrix} A(p) - L_a(p)C & B(p) \\ -F_a(p)C(A(p) - L_a(p)C + I) & -F_a(p)CB(p) \end{bmatrix}$$

$$\tilde{e}_a = \begin{bmatrix} e_x \\ e_{f_a} \end{bmatrix}, \tilde{N}(p,p) = \begin{bmatrix} -L_a(p)D_f & 0 \\ F_a(p)CL_a(p)D_f & I \end{bmatrix}, \tilde{z} = \begin{bmatrix} e_{f_s} \\ \dot{f}_a \end{bmatrix}$$

The objective now is to compute the gains $L_a(p)$ and $F_a(p)$ such that exogenous input $\tilde{z}$ in (29) are attenuated below the desired level $\gamma_a$ to ensure robust regulation performance, in addition to locating the observer poles within a specified LMI $\mathcal{D}(\rho_a, \alpha_a, \beta_a, \theta_a)$ region.

***Theorem 2****. The eigenvalues of the estimation error are located in a disc region in the complex plane defined by*

*($\rho_a, \alpha_a, \beta_a, \theta_a$ ), and the error dynamics are stable and the $H_\infty$ performance is guaranteed with an attenuation level $\gamma_a$, (provided that the signal ($\tilde{z}$) is bounded), if there exists a SPD matrix $P_{a1}$, together with matrices $H_{ai}, F_{ai}$, and scalar parameters $\mu_a$, $\alpha_a$, $\rho_a$, $\theta_a$, and $\beta_a$ satisfying the following LMI constraints*:

*Minimize ($\gamma_a + \mu_a$) such that:*

$$\left.\begin{array}{c} \Sigma_{ai} + \Sigma_{ai}^T + 2\rho_a \bar{P}_a < 0 \\ \begin{bmatrix} -\alpha_a \bar{P}_a & \beta_a \bar{P}_a + \Sigma_{ai} \\ (\beta_a \bar{P}_a + \Sigma_{ai})^T & -\alpha_a \bar{P}_a \end{bmatrix} < 0 \\ \begin{bmatrix} \sin(\theta)\left[\mathbb{X}\mathcal{A} + (\mathbb{X}\mathcal{A})^T\right] & \cos(\theta)\left[\mathbb{X}\mathcal{A} - (\mathbb{X}\mathcal{A})^T\right] \\ \cos(\theta)\left[\mathbb{X}\mathcal{A} - (\mathbb{X}\mathcal{A})^T\right] & \sin(\theta)\left[\mathbb{X}\mathcal{A} + (\mathbb{X}\mathcal{A})^T\right] \end{bmatrix} < 0 \end{array}\right\} \tag{30}$$

$$\begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} & 0 & C_{p1}^T & 0 \\ * & \Psi_{22} & \Psi_{23} & I & 0 & C_{p2}^T \\ * & * & -\gamma_a I & 0 & 0 & 0 \\ * & * & * & -\gamma_a I & 0 & 0 \\ * & * & * & * & -\gamma_a I & 0 \\ * & * & * & * & * & -\gamma_a I \end{bmatrix} < 0 \tag{31}$$

$$\begin{bmatrix} \mu_a I & B(p)^T P_1 - F_a(p) C \\ * & \mu_a I \end{bmatrix} > 0 \tag{32}$$

$$L_a(p) = P_{a1}^{-1} H(p)\,;\Psi_{13} = -H(p) D_f$$

$$\Psi_{11} = P_{a1} A(p) + (P_{a1} A(p))^T - H(p) C - (H(p) C)^T$$

$$\Psi_{12} = -\left(A^T(p) P_{a1} B(p) - C^T H^T(p) B(p)\right)$$

$$\Psi_{22} = -\left(B^T(p) P_{a1} B(p) + B(p) P_{a1} B^T(p)\right)$$

$$\Psi_{23} = -B^T(p) H(p) D_f$$

$$\Sigma_{ai} = \begin{bmatrix} P_{a1} A(p) - \bar{H}(p) C & P_{a1} B(p) \\ \Phi & -B^T(p) P_{a1} B(p) \end{bmatrix}$$

$$\Phi = -\left(A^T(p) P_{a1} B(p) - C^T H^T(p) B(p) + B^T(p) P_{a1}\right)^T$$

**Proof.** This proceeds in a similar manner to steps given to prove *Theorem 1* and hence is omitted. To overcome the equality constraint required throughout the proof of *Theorem 2*, the inequality (23) is also required as given below:

minimise $\mu_a$

$$\begin{bmatrix} \mu_a I & B(p)^T P_1 - F_a(p) C \\ * & \mu_a I \end{bmatrix} > 0 \tag{33}$$

### 3.3 Controller Design

The control objective is to design a TSDOFC capable of forcing the specified output of the nonlinear plant to follow a bounded time-varying reference signal in both the faulty and fault-free cases.

An augmented system consisting of the system (4) and the integral of the tracking error ($e_{ti} = \int y_r - Sy$) is defined below:

$$\left.\begin{array}{l} \dot{\bar{x}} = \bar{A}(p)\bar{x} + \bar{B}(p)(u + f_a) + R y_r + D_{in} e_{f_s} \\ \bar{y} = \bar{C}\,\bar{x} + \bar{D}_f\, e_{f_s} \end{array}\right\} \tag{34}$$

$$\bar{A}(p) = \begin{bmatrix} 0 & -SC \\ 0 & A(p) \end{bmatrix}, \bar{x} = \begin{bmatrix} e_{ti} \\ x \end{bmatrix}, \bar{B}(p) = \begin{bmatrix} 0 \\ B(p) \end{bmatrix}$$

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix}, D_{in} = \begin{bmatrix} -SD_f \\ 0 \end{bmatrix}, \bar{C} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix}, \bar{D}_f = \begin{bmatrix} 0 \\ D_f \end{bmatrix}$$

where $S \in \mathcal{R}^{q*l}$ is used to define which output variable is considered to track the reference signal. Since the system in (34) has a common output matrix (C), the dynamic output feedback controller used to stabilize and perform the tracking objective is of quadratic parameterisation form and defined below:

$$\left.\begin{array}{l} \dot{x}_c = A_c(p,p) x_c + B_c(p)(S_r y_r - \bar{y}) \\ u = C_c(p) x_c + D_c(p)(S_r y_r - \bar{y}) - \hat{f}_a \end{array}\right\} \tag{35}$$

where $x_c$ is the state and $A_c(p,p) \in \mathcal{R}^{(n+q)\times(n+q)}$, $B_c(p) \in \mathcal{R}^{(n+q)\times(l+q)}$, $C_c(p) \in \mathcal{R}^{m\times(n+q)}$, $D_c(p) \in \mathcal{R}^{m\times(l+q)}, R \in \mathcal{R}^{(n+q)\times q}, D_{in} \in \mathcal{R}^{(n+q)\times g}, and\ S_r \in \mathcal{R}^{(l+q)\times q}$ is introduced to match the dimensions of $y_r$ *and* $\bar{y}$. Aggregation of Eq.(34) and Eq.(35) gives the following system:

$$\left.\begin{array}{l} \dot{x}_a = A_a(p,p) x_a + E_a(p,p)\tilde{d} \\ \bar{y} = C_a x_a + D_a \tilde{d} \end{array}\right\} \tag{36}$$

$$A_a(p,p) = \begin{bmatrix} \bar{A}(p) - \bar{B}(p) D_c(p) \bar{C} & \bar{B}(p) C_c(p) \\ -B_c(p)\bar{C} & A_c(p,p) \end{bmatrix}$$

$$E_a = \begin{bmatrix} \bar{B}(p) & D_{in} - \bar{B}(p) D_c(p) \bar{D}_f & R + \bar{B}(p) D_c(p) S_r \\ 0 & -B_c(p)\bar{D}_f & B_c(p) S_r \end{bmatrix}$$

$$x_a = \begin{bmatrix} \bar{x} \\ x_c \end{bmatrix}, \tilde{d} = \begin{bmatrix} e_{f_a} \\ e_{f_s} \\ y_r \end{bmatrix}; C_a = \begin{bmatrix} \bar{C} & 0 \end{bmatrix}; D_a = \begin{bmatrix} 0 & \bar{D}_f & 0 \end{bmatrix}$$

***Theorem 3***. *If the eigenvalues of the closed-loop system Eq.34 are located in the negative complex plane characterised by the LMI region defined by: $\alpha_c$, $\beta_c$, $\rho_c$, $\theta_c$ , then the closed-loop system will be stable. Furthermore, the closed-loop system will track the reference signal with guaranteed $H_\infty$ performance with an attenuation level $\gamma_c$, (provided that the augmented signal $\tilde{d}$ is bounded), if there exist SPD matrices $X, Y$, and matrices $A_c(p,p)$, $B_c(p)$, $C_c(p)$, $D_c(p)$, together with scalars $\alpha_c$, $\rho_c$, $\theta_c$, and $\beta_c$ that satisfy the following LMI constraints:*

*Minimize $\bar{\gamma}_c$ such that:*

$$\left.\begin{array}{c} \mathcal{Q}_{ij} + \mathcal{Q}_{ij}^T + 2\rho_c X < 0 \\ \begin{bmatrix} -\alpha_c X & \beta_c X + \mathcal{Q}_{ij} \\ \beta_c X + \mathcal{Q}_{ij}^T & -\alpha_c X \end{bmatrix} < 0 \\ \begin{bmatrix} \sin(\theta_c)\left[\mathcal{Q}_{ij} + \mathcal{Q}_{ij}^T\right] & \cos(\theta_c)\left[\mathcal{Q}_{ij} - \mathcal{Q}_{ij}^T\right] \\ \cos(\theta_c)\left[\mathcal{Q}_{ij} - \mathcal{Q}_{ij}^T\right] & \sin(\theta_c)\left[\mathcal{Q}_{ij} + \mathcal{Q}_{ij}^T\right] \end{bmatrix} < 0 \end{array}\right\} \tag{37}$$

$$\begin{bmatrix} \Psi_{11c} & \Psi_{12c} & \bar{B}(p) & \Psi_{13c} & \Psi_{14c} & X\bar{C}^T & -X\bar{C}^T & 0 \\ * & \Psi_{22c} & Y\bar{B}(p) & \Psi_{23c} & \Psi_{24c} & \bar{C}^T & -\bar{C}^T & 0 \\ * & * & -\gamma_c I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\gamma_c I & 0 & 0 & 0 & 0 \\ * & * & * & * & S_r^T S_r - \gamma_c I & 0 & 0 & S_r^T \\ * & * & * & * & * & -I & 0 & 0 \\ * & * & * & * & * & * & -G^{-1} & 0 \\ * & * & * & * & * & * & * & -G \end{bmatrix} < 0 \tag{38}$$

where

$$\Psi_{11c} = \bar{A}(p)X + \left(\bar{A}(p)X\right)^T + \bar{B}(p)\hat{C}(p) + \left(\bar{B}(p)\hat{C}(p)\right)^T$$

$$\Psi_{12c} = \hat{A}^T(p,p) + \bar{A}(p) - \bar{B}(p)\hat{D}(p)\bar{C}$$

$$\Psi_{13c} = D_{in} - \bar{B}(p)\hat{D}(p)\bar{D}_f \; ; \Psi_{23c} = YD_{in} + \hat{B}(p)\bar{D}_f$$

$$\Psi_{22c} = Y\bar{A}(p) + \left(Y\bar{A}(p)\right)^T + \hat{B}(p)\bar{C} + \left(\hat{B}(p)\bar{C}\right)^T$$

$$\Psi_{14c} = R + \bar{B}(p)\hat{D}(p)S_r \; ; \Psi_{24c} = YR - \hat{B}(p)S_r$$

$$Q_{ij} = \begin{bmatrix} \bar{A}(p)X + \bar{A}(p)\hat{C}(p) & \bar{A}(p) - \bar{B}(p)\hat{D}(p)\bar{C} \\ \hat{A}(p,p) & Y\bar{A}(p) + \hat{B}(p)\bar{C}(p) \end{bmatrix}$$

The controller gains are thus calculated as follows:

$$D_c(p) = \hat{D}(p)$$

$$C_c(p) = \left(\hat{C}(p) + D_c(p)\bar{C}X\right)M^{-T}$$

$$B_c(p) = N^{-1}\left(-\hat{B}(p) - Y\bar{B}(p)D_c(p)\right)$$

$$A_c(p,p) = N^{-1}[\,\hat{A}(p,p) - Y\left(\bar{A}(p) - \bar{B}(p)\hat{D}(p)\bar{C}\right)X \ldots - Y\bar{B}(p)C_c(p)M^T + NB_c(p)\bar{C}X\,]M^{-T}$$

where $M$ and $N$ satisfy $MN^T = I - XY$

**Proof:** Following the definition of the observer estimation performance objective $e_p$ in (16), the controller robustness against the augmented input ($\tilde{d}$) can be represented by minimising the performance objective below:

$$\frac{\|y_p\|_2}{\|\tilde{d}\|_2} \le \gamma_c = \int_0^\infty y_p^T y_p dt - \gamma_c^2 \int_0^\infty \tilde{d}^T \tilde{d}\, dt \le 0 \tag{39}$$

where $y_p$ is the performance objective variable:

$$y_p = (S_r y_r - \bar{y}) ; y_p^T y_p = (S_r y_r - \bar{y})^T (S_r y_r - \bar{y})$$

$$y_p^T y_p = y_r^T S_r^T S_r y_r - \bar{y}^T S_r y_r - \bar{y}_r^T S_r^T \bar{y} + \bar{y}^T \bar{y}$$

Let $E_r = [0\ 0\ S_r]$ then:

$$y_p^T y_p = \tilde{d}^T E_r^T E_r \tilde{d} - \tilde{d}^T E_r^T C_a x_a - \ldots x_a^T C_a^T E_r \tilde{d} + x_a^T C_a^T C_a x_a$$

Consider the following candidate Lyapunov function for the augmented system (36)

$$\upsilon(x_a) = x_a^T \bar{P} x_a \;\; , where\ \bar{P} > 0$$

As stated in the observer design, to achieve the required performance (39) and stability of the augmented system Eq.(36) the following inequality should hold:

$$\dot{\upsilon}(x_a) + y_p^T y_p - \gamma_c^2 \tilde{d}^T \tilde{d} < 0 \tag{40}$$

where $\dot{\upsilon}(x_a)$ is the derivative of the candidate Lyapunov function, based on the state-space representation of the augmented system Eq.(36), inequality (40) then becomes:

$$\dot{\upsilon}(x_a) = x_a^T \left(A_a^T(p,p)\bar{P} + \bar{P}A_a(p,p)\right)x_a \ldots + x_a^T \bar{P}E_a(p)\tilde{d} + \tilde{d}^T E_a^T(p)\bar{P}x_a \tag{41}$$

By using Eq. (41) and the Schur Complement Theorem, inequality (40) implies that the following inequality must hold:

$$\begin{bmatrix} A_a^T(p,p)\bar{P} + \bar{P}A_a(p,p) & \bar{P}E_a(p) - C_a^T E_r & \bar{C}_a^T \\ * & E_r^T E_r - \gamma_c^2 I & 0 \\ * & * & -I \end{bmatrix} < 0 \tag{42}$$

Inequality (42) can further decomposed as below:

$$\begin{bmatrix} A_a^T(p,p)\bar{P} + \bar{P}A_a(p,p) & \bar{P}E_a(p) & \bar{C}_a^T \\ * & E_r^T E_r - \gamma_c^2 I & 0 \\ * & * & -I \end{bmatrix} + \ldots \begin{bmatrix} -C_a^T \\ 0 \\ 0 \end{bmatrix} [0\ E_r\ 0] + \begin{bmatrix} 0 \\ E_r^T \\ 0 \end{bmatrix} [-C_a\ 0\ 0] < 0 \tag{43}$$

***Lemma 2*** [17]*: Given a scalar* $\mu > 0$ *and SPD matrix G, the following inequality holds:*

$$X^T R + R^T X \le X^T G X + R^T G^{-1} R \tag{44}$$

where $R$ & $X$ are two matrices.

Based on Lemma 2, inequality (43) is implied by the following inequality:

$$\begin{bmatrix} \Delta & \bar{P}E_a(p) & \bar{C}_a^T & -C_a^T & 0 \\ * & E_r^T E_r - \gamma_c^2 I & 0 & 0 & E_r^T \\ * & * & -I & 0 & 0 \\ * & * & * & -G^{-1} & 0 \\ * & * & * & * & -G \end{bmatrix} < 0 \tag{45}$$

$$\Delta = A_a^T(p,p)\bar{P} + \bar{P}A_a(p,p)$$

It can be assumed that $\bar{P}$ and $\bar{P}^{-1}$ is structured as follows:

$$\bar{P} = \begin{bmatrix} Y & N \\ N^T & * \end{bmatrix}, \bar{P}^{-1} = \begin{bmatrix} X & M \\ M^T & * \end{bmatrix}, \text{since } \bar{P}\bar{P}^{-1} = I$$

we then have $\bar{P}\begin{bmatrix} X \\ M^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \Rightarrow \bar{P}\begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix} = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}$.

Define $\Pi_1 = \begin{bmatrix} X & I \\ M^T & 0 \end{bmatrix}$ ; $\Pi_2 = \begin{bmatrix} I & Y \\ 0 & N^T \end{bmatrix}$

Pre- and post-multiplying inequality (45) by $[\Pi_1^T \quad I \quad I \quad I \quad I]$ and its transpose respectively, the following inequality obtained:

$$\begin{bmatrix} \Delta_\Pi & \Pi_2^T E_a(p) & \Pi_1^T \bar{C}_a^T & -\Pi_1^T \bar{C}_a^T & 0 \\ * & E_r^T E_r - \gamma_c^2 I & 0 & 0 & E_r \\ * & * & -I & 0 & 0 \\ * & * & * & -G^{-1} & 0 \\ * & * & * & * & -G \end{bmatrix} < 0 \tag{46}$$

$$\Delta_\Pi = \Pi_1^T A_a^T(p,p)\Pi_2 + \Pi_2^T A_a(p,p)\Pi_1$$

After simple algebraic manipulation and using the following change of variables:

$$\begin{aligned} \hat{A}(p,p) &= Y\left(\bar{A}(p) - \bar{B}(p)D_c(p)\bar{C}\right)X + \ldots \\ &\quad Y\bar{B}(p)C_c(p)M^T - NB_c(p)\bar{C}X + NA_c(p,p)M^T \\ \hat{B}(p) &= -NB_c(p) - Y\bar{B}(p)D_c(p) \\ \hat{C}(p) &= C_c(p)M^T - D_c(p)\bar{C}X \\ \hat{D}(p) &= D_c(p) \\ \bar{\gamma}_c &= \gamma_c^2 \end{aligned}$$

Then inequality (38) can be obtained easily.

By using the change of variables $A = A_a(p,p)$, $X = \bar{P}$, and pre- and post-multiplying the first inequality of (23) by $\Pi_1^T$, the 2nd and 3rd inequality of (24) by $[\Pi_1^T \quad \Pi_1^T]$ and its transpose respectively yields:

$$\left.\begin{array}{c} \Im_+ + 2\rho \Pi_1^T \Pi_2 < 0 \\ \begin{bmatrix} -\alpha \Pi_1^T \Pi_2 & \beta \Pi_1^T \Pi_2 + (\Im_+ + \Im_-)/2 \\ \beta \Pi_1^T \Pi_2 + (\Im_+ + \Im_-)^T/2 & -\alpha \Pi_1^T \Pi_2 \end{bmatrix} < 0 \\ \begin{bmatrix} \sin(\theta)[\Im_+] & \cos(\theta)[\Im_-] \\ \cos(\theta)[\Im_-] & \sin(\theta)[\Im_+] \end{bmatrix} < 0 \end{array}\right\}$$

where : (47)

$$\Im_+ = \Pi_2^T A_a(p,p)\Pi_1 + \left(\Pi_2^T A_a(p,p)\Pi_1\right)^T$$
$$\Im_- = \Pi_2^T A_a(p,p)\Pi_1 - \left(\Pi_2^T A_a(p,p)\Pi_1\right)^T$$

Substituting the equality $MN^T = I - XY$ into inequality (47), then inequality (37) is obtained.

***Remark 4:*** The matrices $M, N^T$ can be determined based on the equality $MN^T = I - XY$ using any matrix decomposition techniques e.g. QR (*qr*) decomposition or Singular Value Decomposition (*svd).*

***Remark 5:***

- The proposed methodology offers design freedom to combine any estimation strategy for actuator and sensor faults. Moreover, the time responses of the two fault estimation observers as well as the closed-loop control system can be adjusted separately.
- Due to the fact that T-S fuzzy static output feedback controller (SOFC) has a non convex Lyapunov stability condition [31], in this paper the fuzzy DOFC (TSDOFC) is proposed instead of SOFC.

## 4. INVERTED PENDULUM EXAMPLE

To illustrate the proposed FTC strategy encompassing the possibility of simultaneous actuator and sensor faults, a tutorial example is considered using a nonlinear simulation of the inverted pendulum and cart with tracking of a time-varying reference cart position. As the pendulum system is nonlinear a local approximation-based T-S fuzzy model has been derived based on the procedure given in [32]. The faults considered are additive and parametric as follows. Various results are generated by considering the cart position sensor and cart actuator to have both additive and parametric faults.

The nonlinear inverted pendulum and cart system model is given as follows:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ \dfrac{g\sin(x_1)}{4l/3 - mla(\cos(x_1))^2} \\ \dfrac{-mag\sin(2x_1)/2}{4/3 - mla(\cos(x_1))^2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{-a\cos(x_1)}{4l/3 - mla(\cos(x_1))^2} \\ \dfrac{4a/3}{4/3 - mla(\cos(x_1))^2} \end{bmatrix} \ldots$$
$$* \left(u + \left(f_a + mlx_3^2 \sin(x_1)\right)\right) \tag{48}$$

where $x_1$, $x_2$ $x_3$, $x_4$ are the pendulum angle position, the cart position, the pendulum angular velocity, and the cart speed, respectively. The system parameters are $m$ : Pendulum mass (2kg), $2l$: Pendulum length (1m), $M$: Cart mass (8kg), $a = \frac{1}{m+M}$. The output matrix is:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Although, increasing the number of fuzzy rules ensures good approximation of a smooth nonlinear system, the design conservatism of the T-S fuzzy controller and estimator also increase. Therefore, to take into account this trade-off, three system operating points are chosen corresponding to the pendulum angular positions $\theta = 0$ *and* $\pm \pi/4$ . Due to symmetry this results in the choice of *two* fuzzy rules in the T-S model. Details of the fuzzy model are presented in [32] and are omitted here. The control objective is to force the cart position to follow a desired cart reference position in the presence of a cart position measurement $C_2 x$ fault and actuator fault.

By solving the LMI conditions given in Theorems 1, 2, and 3 the fuzzy controller and observers gains are:

$$A_{c(1,1)} = \begin{bmatrix} -1.49 & 2.01 & 10.35 & -29.38 & 806.95 \\ -1.10 & -2.23 & 7.67 & -20.95 & 591.71 \\ 0.34 & -0.27 & -2.56 & 6.30 & -187.19 \\ -0.09 & 0.11 & 0.11 & -2.95 & 39.95 \\ 0.09 & -0.36 & -0.38 & 0.73 & -17.16 \end{bmatrix}$$

$$A_{c(1,2)} = \begin{bmatrix} -1.93 & 0.96 & 17.18 & -61.83 & 2748.86 \\ -1.43 & -3.01 & 12.72 & -44.92 & 2027.57 \\ 0.45 & -0.02 & -4.15 & 13.89 & -637.38 \\ -0.12 & 0.06 & 0.44 & -4.51 & 128.92 \\ 0.10 & -0.34 & -0.51 & 1.31 & -52.15 \end{bmatrix}$$

$$A_{c(2,1)} = \begin{bmatrix} -1.46 & 2.26 & 10.19 & -28.09 & 752.52 \\ -1.12 & -2.01 & 5.23 & -13.11 & 333.50 \\ 0.33 & -0.33 & -2.37 & 5.50 & -161.40 \\ -0.08 & 0.12 & 0.11 & -2.90 & 42.72 \\ -0.04 & -0.34 & -0.20 & 0.27 & -2.24 \end{bmatrix}$$

$$A_{c(2,2)} = \begin{bmatrix} -1.88 & 1.29 & 16.49 & -58.02 & 2543.07 \\ -1.30 & -2.42 & 7.89 & -25.74 & 1086.62 \\ 0.42 & -0.12 & -3.70 & 11.88 & -538.60 \\ -0.10 & 0.07 & 0.43 & -4.40 & 127.20 \\ -0.04 & -0.34 & -0.22 & 0.36 & -8.81 \end{bmatrix}$$

$$B_{c1} = \begin{bmatrix} 12.27 & -51.36 & -334.10 & -131.49 \\ 8.64 & -11.49 & -251.91 & 22.22 \\ 45.03 & -0.04 & 49.63 & 118.87 \\ 141.63 & 20.10 & -111.01 & -37.84 \\ 17.20 & -194.45 & -4.07 & 7.94 \end{bmatrix}$$

$$B_{c2} = \begin{bmatrix} 5.34 & -27.38 & -330.13 & -132.20 \\ 2.29 & -8.29 & -146.23 & 17.92 \\ 46.67 & -5.53 & 41.87 & 118.13 \\ 141.26 & 21.27 & -111.55 & -36.10 \\ 16.69 & -182.50 & -9.54 & -9.34 \end{bmatrix}$$

$$C_{c1} = [-0.02 \quad 0.17 \quad 0.75 \quad -2.19 \quad 59.86]$$
$$C_{c2} = [-0.04 \quad 0.10 \quad 1.27 \quad -4.59 \quad 203.37]$$
$$D_{c1} = [0.74 \quad -2.78 \quad -24.56 \quad -0.07]$$
$$D_{c2} = [0.26 \quad -1.62 \quad -26.30 \quad -0.01]$$

The sensor fault T-S PPI gains are calculated as:

$$\bar{L}_1 = \begin{bmatrix} 39.32 & -0.26 & -8.79 \\ -1.71 & 0.02 & 0.46 \\ 142.99 & -1.02 & -32.15 \\ -5.99 & 0.25 & 1.86 \\ 33.28 & -0.22 & -7.43 \\ -13.71 & -17.88 & 4.19 \\ -4.65 & 0.24 & 1.55 \end{bmatrix}, \bar{L}_2 = \begin{bmatrix} 40.45 & -0.27 & -1.24 \\ -1.82 & 0.02 & 0.11 \\ 147.14 & -1.05 & -4.70 \\ -6.23 & 0.25 & 0.70 \\ 34.24 & -0.22 & -1.04 \\ -15.20 & -17.88 & 1.13 \\ -4.86 & 0.24 & 0.65 \end{bmatrix}$$

The actuator fault T-S PPI gains are calculated as:

$$L_1 = \begin{bmatrix} 522.50 & -0.04 & -1.42 \\ 0.05 & 1.61 & 0.99 \\ 769.44 & 0.08 & -3.28 \\ 0.01 & 0.04 & 1.99 \end{bmatrix}, L_2 = \begin{bmatrix} 522.54 & -0.06 & -1.15 \\ 0.05 & 1.61 & 0.99 \\ 768.29 & 0.09 & -0.03 \\ 1.99 & 0.08 & 1.75 \end{bmatrix}$$

The fault estimation observer feedback gains for the sensor and actuator PPIs are

$F_s = [12.33 \;\; 7.10 \;\; 2.61]$ $F_a = [-23.48 \;\; -0.01 \;\; 546.39]$

The corresponding attenuation coefficients are $\gamma_c = 0.6302$, $\gamma = 0.2722$ and $\gamma_a = 0.1227$.

The controller designed LMI region is bounded by $\alpha_c = 20$, $\beta_c = 0$, $\rho_c = 0$, $\theta_c = \pi/3$, the sensor fault PPI LMI region is bounded by $\alpha = 100$, $\beta = 0$, $\rho = -1$, $\theta = \pi/2$, and the actuator fault PPI LMI region is bounded by $\alpha_a = 100$, $\beta_a = 0$, $\rho_a = -1$, $\theta_a = \pi/2$.

***Remark 6:*** Although specifying the closed-loop performance via additional LMI constraints can guarantee bounded performance since only a region in the complex plane is defined, this increases the probability of infeasible solutions. Hence, the LMI regions parameters ($\alpha_c$, $\rho_c$, $\theta_c$, $\beta_c$, $\alpha_a$, $\rho_a$, $\theta_a$, $\beta_a$, $\alpha$, $\rho$, $\theta$, *and* $\beta$) should be selected in order to jointly achieve acceptable performance and robustness (i.e. feasibility of inequalities 14, 31, and 38 with minimum $\gamma_c, \gamma_a$ , and $\gamma$ ).

Fig. 2 a, b, c & d show the actuator fault estimation results generated via PPI T-S fuzzy observers, covering several additive fault scenarios of abruptly varying amplitudes and slow to fast (linear time-varying fault frequencies).

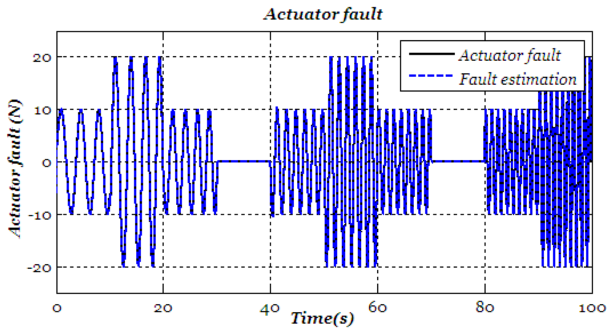


Fig. 2a Time-varying actuator fault signal and fault estimation

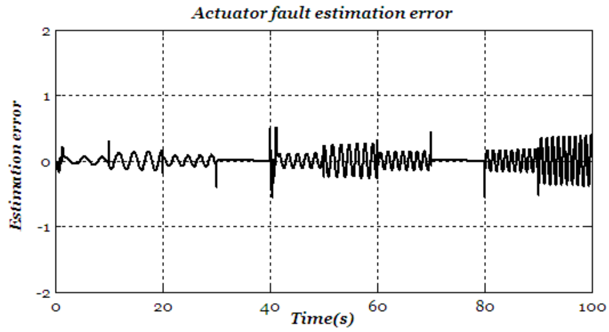


Fig. 2b: Actuator fault estimation error

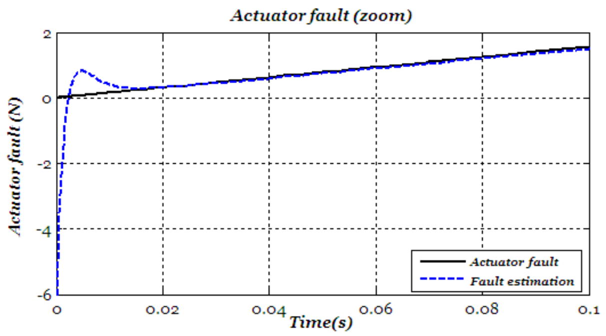


Fig. 2c: Time axis zoomed-in actuator fault signal/estimation

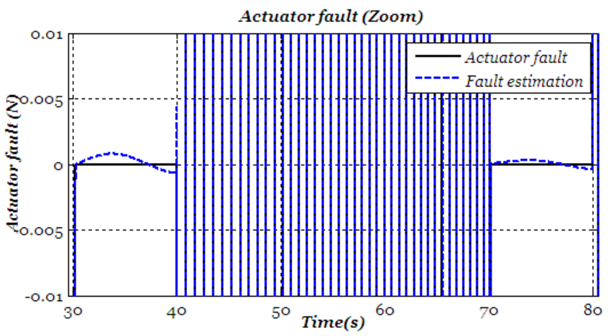


Fig. 2d: Fault magnitude zoomed-in actuator fault signal/estimation

The following simulation results consider the effect of the actuator fault given in Fig.2a with online estimation (via one T-S PPI observer) and compensation. Additive and parametric cart position sensor fault scenarios have been introduced to show the ability of the proposed strategy to handle simultaneous faults. In Fig. 3 a parametric change on the cart position sensor fault

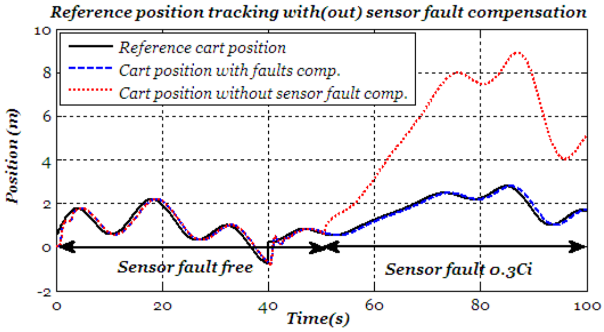


Fig. 3 Simultaneous actuator and sensor fault uncompensated sensor fault

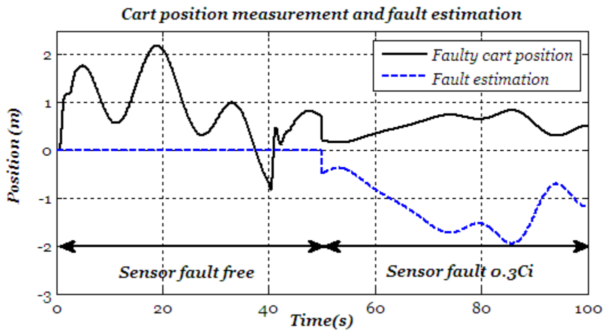


Fig. 4 Faulty measurement and fault estimatio

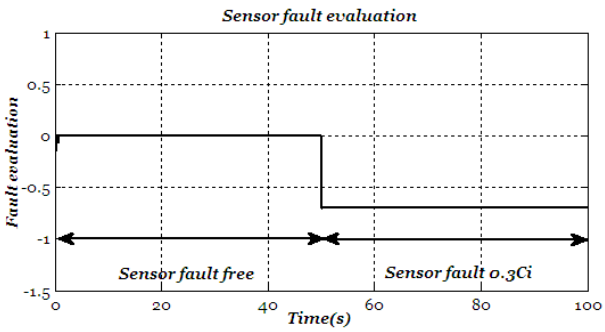


Fig. 5 Sensor fault evaluation

($0.3C_2$) is introduced and the proposed FTTC system maintains the tracking performance during the simultaneous fault.

Fig. 4 shows the effectiveness of the actuator fault compensation. The significant effect of the uncompensated sensor fault is also shown. Moreover, the fault estimation in Fig. 4 indicates that a parametric change fault is a special case of an additive fault in which the fault signal represents the loss of effectiveness multiplied by the corresponding fault-free signal. Based on this interpretation, the fault estimation signal can be utilized to assess the severity of the fault as shown in Fig. 5. This is achieved by taking the ratio between the measured cart position and the faulty estimation signal. Hence, if there are no faults the ratio should be 0 otherwise any deviation indicates the occurrence of the fault and the magnitude of the deviation represents the fault severity.

Fig. 6 a & b shows a result from further investigation of the proposed FTTC system by considering a time-varying and abruptly changing multi-step sensor fault signal and its T-S PPI estimate affecting the system at the same time as the actuator fault shown in Fig. 2a. Zoomed version of the sensor fault estimate signal in Fig. 6 c & d with separate time windows further demonstrate the effectiveness of the proposed estimation and compensation scheme. Within each window small sinusoidal variations from the actuator fault (with different frequencies and amplitude) are clearly visible showing that the bi-directional interactions between the proposed T-S PPI observers are strongly attenuated.

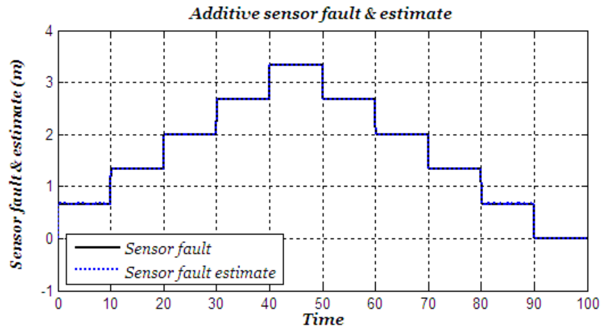


Fig. 6 a: Additive sensor fault signal/estimation

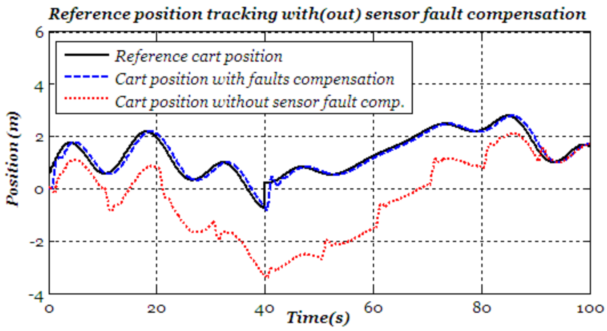


Fig. 6 b: Simultaneous actuator & sensor fault with the uncompensated sensor fault

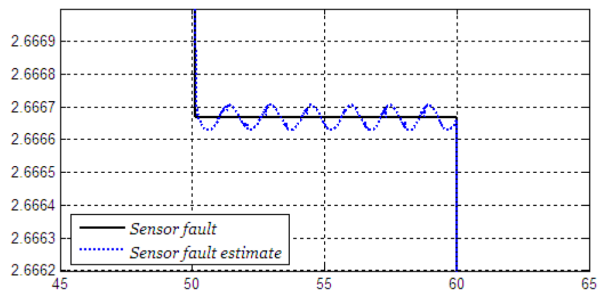


Fig. 6 c: Zoomed-in sensor fault signal/estimation

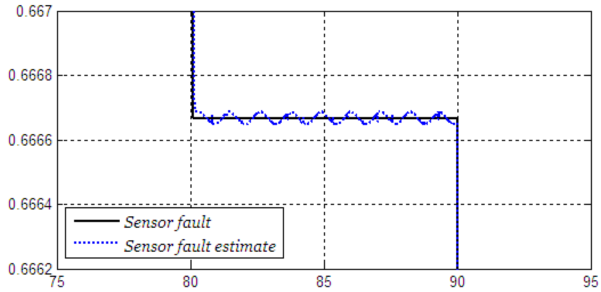


Fig. 6 d: Zoomed-in sensor fault signal/estimation

## 5. CONCLUDING DISCUSSION

The paper develops a new architecture for active FTC with simultaneous actuator and sensor faults based on fault estimation and compensation. Using this proposed architecture, a detailed design approach is presented for a certain class of nonlinear systems that can be modelled in T-S fuzzy inference form. A dynamic output feedback control scheme is used that has time-varying reference tracking capability and the fault estimators are designed using a proposed T-S extension of a well known PPI observer scheme, the T-S PPI observer. The controller and fault estimators individually satisfy an appropriate $L_2$ norm robustness condition guaranteeing minimum tracking error and robust fault estimation. From this a dual pair of actuator and sensor fault estimators are developed that have low fault interaction and which together provide robust fault compensation in the output feedback controller. A tutorial study based on a nonlinear inverted pendulum shows how the proposed FTTC can handle the most challenging and complex FTC problem, that of simultaneous actuator and sensor faults.

In summary, the significant attributes gained by using the FTTC system are:

1. Design of an FTTC system that robustly tolerates simultaneous sensor and actuator faults.
2. Estimate time-varying actuator and sensor faults with bounded first time-derivatives using proportional and integral feedback PPI observers with T-S model structure.
3. Maintain the nominal controller performance in the presence of large reference changes and faults.
4. Overcome the hurdles imposed by the generally accepted use of T-S observer-based state feedback. Moreover, the limitation of using an iterative form of static output feedback control design is obviated via the use of TSDOFC
5. The significant impact of tracking control for the sensor fault tolerance problem is investigated.

Furthermore, the investigation has also shown that additive faults are a generalized fault representation that can be used to assess the severity of sensor faults. These factors represent significant contributions to the subject in active FTC.

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